Coefficient scaling

Gerald Paul*
43 Highland Avenue, Lexington, Massachusetts 02421
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We prove a remarkably simple but powerful recursion relation for the coefficients of iterated polynomials. We also prove that the recursion relation holds for the coefficients of certain functions of the iterated polynomial. Using the recursion relations, we obtain a closed-form expression for the average number of closed-loop self-avoiding walks per site on a family of fractal lattices. We describe numerical results, which exhibit log-periodic oscillations, and find good agreement between these results and the theory developed here, which predicts the existence of the log-periodic oscillations and their amplitudes. Finally, we discuss insights gained into the mathematical origins of critical phenomena. [S1063-651X(99)10704-9]

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I. INTRODUCTION

Recursion relations involving polynomials or rational functions play a major role in a number of branches of physics. In many cases, the function involved is the renormalization group transformation applicable to the system being investigated [1]. Usually this function is linearized around a fixed point x_c and the recursion relation is then applied to this linearized equation. In doing this, information which is needed to calculate the amplitudes of the quantities exhibiting critical behavior is lost. Also there are situations in which the linearization results in unphysical divergences.

We have discovered and proven a recursion relation for the coefficients of iterated polynomials. Using this recursion relation, we formulate an approach to calculating physical observables which exhibit critical behavior; the approach employs the coefficients of the renormalization group transformation function as opposed to the function itself. This approach has no divergences and allows the amplitudes of the critical quantities to be calculated straightforwardly.

In Sec. VI, we use this technique to study the critical behavior of p_n, the average number of closed-loop self-avoiding walks per site on the family of Sierpinski gaskets.

Finally, we discuss how the properties of polynomials can be used to understand the mathematical origins of critical phenomena.

II. RECURSION RELATION

Consider a polynomial of degree <1,

$$F(x)=\sum_n a_n x^n, \tag{2.1}$$

with non-negative coefficients, with fixed point $x_c>0$, defined by $F(x_c)=x_c$, connectivity constant, $\mu\equiv 1/x_c$, and corresponding eigenvalue

$$\lambda\equiv\frac{d}{dx}F(x)\bigg|_{x_c}>1.$$

We define the rth iteration of $F(x)$ as

$$F^{(r)}(x)\equiv\sum a_n^{(r)}x^n\equiv F(F^{(r-1)}(x)). \tag{2.2}$$

We will prove that for large r,

$$\frac{a_n^{(r)}}{\mu^n}=\frac{1}{\lambda}\,\frac{a_{n/\lambda}^{(r-1)}}{\mu^{n/\lambda}}. \tag{2.3}$$

We call this property coefficient scaling. In what follows we will refer to the coefficients of a polynomial divided by μ^n as normalized coefficients. When n/λ is not an integer, $a_{n/\lambda}$ will be understood to be an interpolated value. Also, if $F(x)$ is a function odd or even in x, the recursion relation is understood to apply to the nonzero odd or even terms of $F^{(r)}(x)$.

A. Approach

The proof will proceed by (i) deriving a closed-form expression for the derivatives of $F^{(r)}(x)$ at x_c, (ii) deriving a closed-form expression for the jth moment of the normalized coefficients, and (iii) showing that the corresponding moments of the normalized coefficients, $p_n^{(r)}\equiv a_n^{(r)}/\mu^n$ and $q_n^{(r)}\equiv\lambda^{-1}a_{n/\lambda}^{(r-1)}/\mu^{n/\lambda}$, are equal.

B. Behavior of *j*th derivative

Using the chain rule of differentiation, we see that

$$\lambda^{(r)}\equiv\frac{d}{dx}(F^{(r)}(x))\bigg|_{x_c}=\lambda\frac{d}{dx}(F^{(r-1)}(x))\bigg|_{x_c}$$

and then by induction that

$$\lambda^{(r)}=\lambda^r.$$

Similarly,

*Present address: Center for Polymer Studies & Department of Physics, Boston University, Boston, MA 02215. Electronic address: gerry@argento.bu.edu

$$\sigma^{(r)} \equiv \frac{d^2}{dx^2} F^{(r)}(x)\bigg|_{x_c} = \frac{d}{dx}\left[F(F^{(r-1)}(x))\frac{d}{dx}(F^{(r-1)}(x))\right]\bigg|_{x_c}$$

$$= \sigma\lambda^{2(r-1)} + \lambda\frac{d^2}{dx^2}(F^{(r-1)}(x)),$$

where $\sigma \equiv d^2F(x)/dx^2$, and by induction,

$$\sigma^{(x)} = \sigma\lambda^{2r}\sum_{n=2}^{r-1}\lambda^{-n} = \sigma\lambda^{2r}\frac{(1-\lambda^{2-r})}{\lambda(\lambda-1)}.$$

Since $|\lambda|>1$, we have, for large r,

$$\sigma^{(r)} = \frac{\sigma\lambda^{2r}}{\lambda(\lambda-1)}.$$

Proceeding as before, we have, after some algebra,

$$\tau^{(r)} \equiv \frac{d^3}{dx^3}F^{(r)}(x)\bigg|_{x_c} = \tau\lambda^{3(r-1)} + \frac{3\sigma^2\lambda^{2(r-1)}}{(\lambda-1)\lambda} + \lambda\tau^{(r-1)},$$

where $\tau \equiv d^3F(x)/dx^3$. For large r, we can ignore the second term and this recursion relation simplifies to

$$\tau^{(r)} = \tau\lambda^{3(r-1)} + \lambda\tau^{(r-1)}.$$

By induction,

$$\tau^{(r)} = \tau\lambda^{3r}(\lambda^{-3}+\lambda^{-5}+\lambda^{-7}+) = k_3\lambda^{3r},$$

where k_3 is a constant the value of which will not be important to us.

Looking at the structure of the equations above, we see that for arbitrary j,

$$\frac{d^j}{dx^j}(F^{(r)}(x))\bigg|_{x_c} = \left[\frac{d^j}{dx^j}(F(x))\right]\bigg|_{x_c}\lambda^{j(r-1)} + O(\lambda^{(j-1)(r-1)}) + \lambda\frac{d^j}{dx^j}(F^{(r-1)}(x))\bigg|_{x_c}.$$

Ignoring the terms of lower order in λ, we have by induction,

$$\frac{d^j}{dx^j}(F^{(r)}(x))\bigg|_{x_c} = k_j\lambda^{jr}, \tag{2.4}$$

where the k_j are constants.

This intermediate result in the proof is interesting in its own right. If we define

$$D(r,j) \equiv \left(\frac{d^j}{dx_j}\right)F^{(r)}(x)\bigg|_{x_c},$$

we see that $D(r,j)$ scales in the sense that

$$D(\alpha r,\beta j) = (D(r,j))^{\alpha\beta}.$$

C. Behavior of the *j*th moment

We denote the jth moment of the normalized coefficients a_n/μ^n of F as defined in Eq. (2.1) as

$$M_j(F) = \sum_n n^j\frac{a_n}{\mu^n}.$$

We will show that for large r and polynomials for which $|\lambda|>1$,

$$M_j(F^{(r)}) = x_c^j\frac{d^j}{dx^j}(F^{(r)}(x))\bigg|_{x_c}. \tag{2.5}$$

Consider

$$\frac{d^j}{dx^j}(F^{(r)}(x))\bigg|_{x_c}$$

$$= \sum_{n=0}^{a_{\max}^{(r)}}[n(n-1)(n-2)\dots(n-j+1)]a_n^{(r)}x_c^{n-j}$$

$$= x_c^{-j}\sum_{n=0}^{a_{\max}^{(r)}}[n(n-1)(n-2)\dots(n-j+1)]a_n^{(r)}x_c^n$$

$$= x_c^{-j}[M_j(F^{(r)}) + O(M_{k<j}(F^{(r)}))].$$

Using Eq. (2.4), we have

$$M_j(F^{(r)}) = k_j(x_c\lambda^r)^j - O(M_{k<j}(F^{(r)}))]. \tag{2.6}$$

For $j=1$, there are no moments of lower order than j in Eq. (2.6) so we can ignore the last term.

We can then proceed by induction and, for each successive j, ignore the terms containing moments of lower order in j because they will be proportional to lower-order derivatives. Thus, for all j,

$$M_j(F^{(r)}) = x_c^j\frac{d^j}{dx^j}(F^{(r)}(x))\bigg|_{x_c} = k_j(x_c\lambda^r)^j. \tag{2.7}$$

D. Proof of recursion relation by equating moments

The last step of the proof is to show that all moments of

$$p_n^{(r)} \equiv \frac{a_n^{(r)}}{\mu^n}$$

are equal to the respective moments of

$$q_n^{(r)} \equiv \frac{1}{\lambda}\frac{a_{n/\lambda}^{(r-1)}}{\mu^{n/\lambda}}.$$

As noted earlier, if $F(x)$ is odd or even in x, it is understood that we are proving the equality for the odd or even terms. This, the requirement that all coefficients of $F(x)$ are nonnegative, and the requirement that the degree of F is greater than 1 ensures that $p_n^{(r)}$ and $q_n^{(r)}$ are smoothly varying functions of n. Consider the jth moment of $q^{(r)}$,

$$M_j(q^{(r)}) = \sum_n n^j\frac{1}{\lambda}\frac{a_{n/\lambda}^{(r-1)}}{\mu^{n/\lambda}}.$$

Substituting $m=n/\lambda$ yields

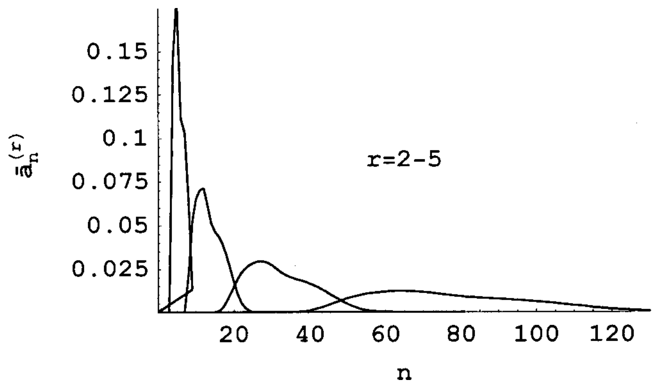

FIG. 1. The normalized coefficients $\overline{a_n^{(r)}}$ for iterations of $F(x)=x^2+x^3$ for $r=2-5$.

$$M_j(q^{(r)})=\frac{1}{\lambda}\sum_{m=0/\lambda,1/\lambda,\ldots}(m\lambda)^j\frac{a_m^{(r-1)}}{\mu^m}.$$

Because the function being summed over is smooth, we can keep only $1/\lambda$ of the terms and multiply by λ. Then,

$$M_j(q^{(r)})=\sum_m (m\lambda)^j\frac{a_m^{(r-1)}}{\mu^m}=\lambda^j M_j(p^{(r-1)}).$$

Using Eq. (2.7), we have

$$M_j(q^{(r)})=\lambda^j k_j(x_c\lambda^{r-1})^j=k_j(x_c\lambda^r)^j=M_j(p^{(r)}).$$

Thus we have shown that the respective moments of the two functions $p_n^{(r)}$ and $q_n^{(r)}$ are equal and, therefore, the functions are equal. That is,

$$\frac{a_n^{(r)}}{\mu^n}=\frac{1}{\lambda}\frac{a_{n/\lambda}^{(r-1)}}{\mu^{n/\lambda}}.$$

Defining $\overline{a_n^{(r)}}=a_n^{(r)}/\mu^n$, we have by induction for $r_0<r$,

$$\overline{a_n^{(r)}}=\left(\frac{1}{\lambda}\right)^{r-r_0}\overline{a_{n/\lambda^{r-r_0}}^{(r-r_0)}}, \tag{2.8}$$

which we will use in Sec. VI.

In Appendix A, we discuss the rate of convergence of the recursion relation.

III. EXAMPLES

In Figs. 1–3, for some simple functions, we plot the normalized coefficients $a_n^{(r)}/\mu^n$ for multiple r. We see that each succeeding plot is lowered in height by $1/\lambda$ and stretched by a factor of λ; the ''area under the curve'' stays constant. As we will see below, this simple geometric property of an iterated polynomial when the polynomial is the renormalization group transformation function explains the exponential be-

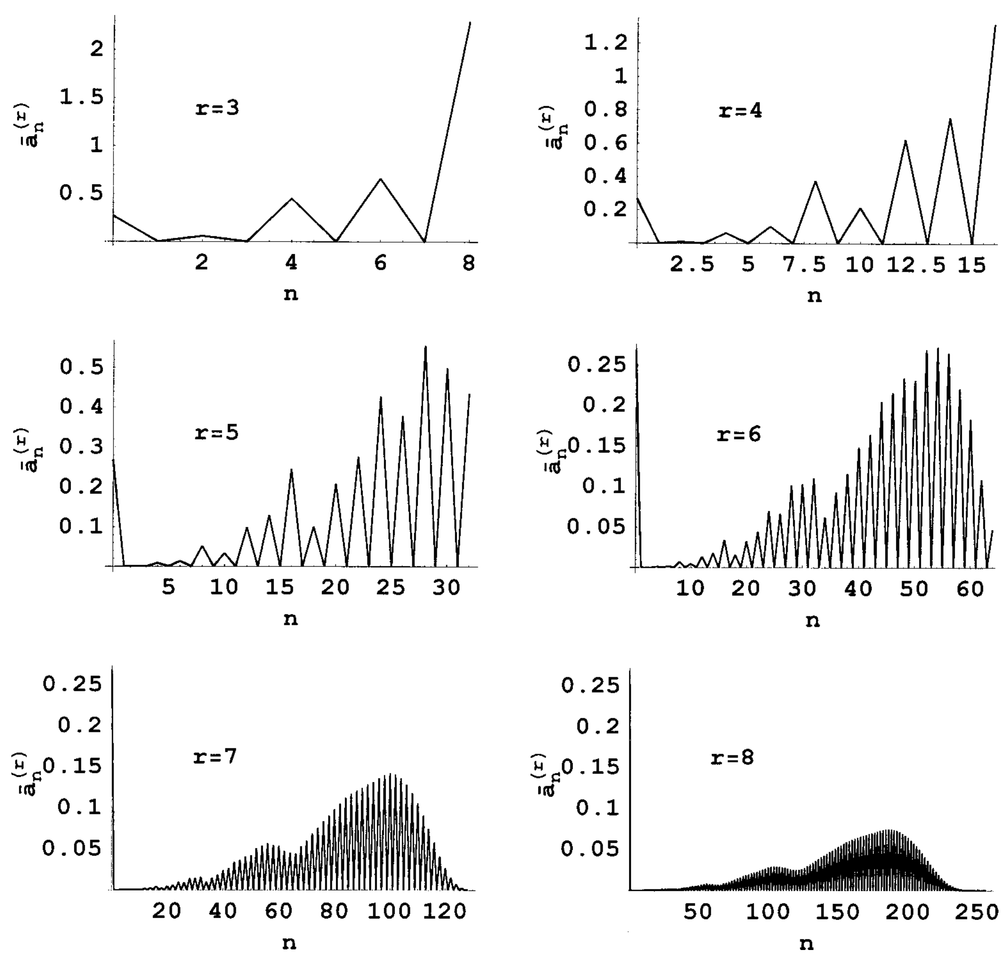

FIG. 2. The normalized coefficients $\overline{a_n^{(r)}}$ for iterations of $F(x)=(1+x^2)/4$ for $r=3-8$.

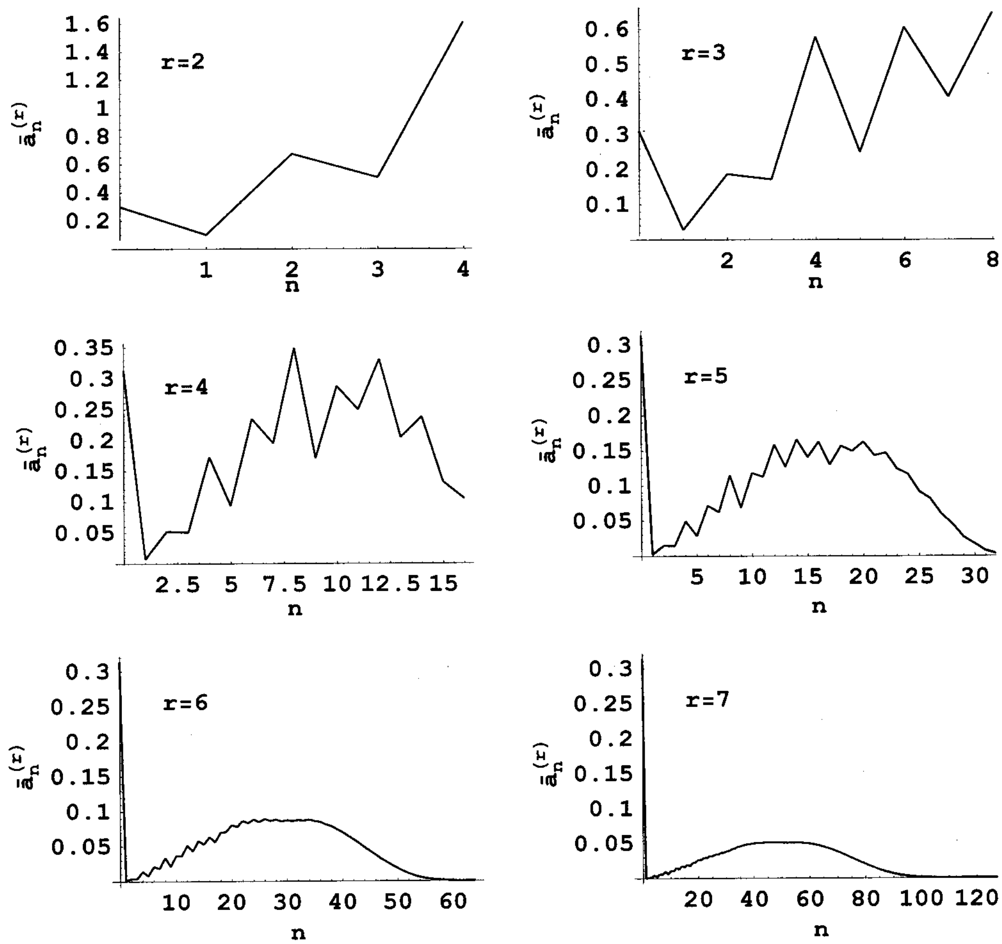

FIG. 3. The normalized coefficients $\overline{a_n^{(r)}}$ for iterations of $F(x)=(1+0.5x+x^2)/4$ for $r=2-7$.

havior and log periodic oscillations of critical phenomena.

The fact that the area under the curve stays constant has to be the case since

$$\sum_n \frac{a_n^{(r)}}{\mu^n}=x_c$$

is true for all r, not just for $r=0$, in which case it is the definition of the fixed point. For higher r, the equation is a sum rule, which constrains the normalized coefficients. The proof of this sum rule is given in Appendix B where we prove a more general sum rule for the normalized coefficients of rational functions of which polynomials are a special case.

IV. SCALING PROPERTIES OF POLYNOMIALS OF POLYNOMIALS

In this section we will show that if the coefficients of $F^{(r)}$ scale as in Eq. (2.3), then the coefficients of polynomials of $F^{(r)}$ also scale in accordance with Eq. (2.3). That is, if H is a polynomial,

$$H(x)\equiv\sum_n h_n x^n$$

and

$$P(x)\equiv H(F(x))\equiv\sum_n p_n x^n$$

$$P^{(r)}(x)\equiv H(F^{(r)}(x))\equiv\sum_n p_n^{(r)} x^n;$$

then,

$$\frac{p_n^{(r)}}{\mu^n}=\frac{1}{\lambda}\frac{p_{n/\lambda}^{(r-1)}}{\mu^{n/\lambda}}. \tag{4.1}$$

To prove this, we first show that if $F^{(r+1)}$ and $G^{(r+1)}$ are two polynomials whose coefficients obey Eq. (2.3), then the coefficients of $(F(x)G(x))^{(r)}$ obey Eq. (2.3). That is, if

$$W(x)\equiv F(x)G(x)\equiv\sum_n w_n x^n,$$

$$W^{(r)}(x)\equiv F^{(r)}(x)G^{(r)}(x)\equiv\sum_n w_{\max}^{(r)}\, w_n^{(r)} x^n;$$

then,

$$\frac{w_n^{(r)}}{\mu^n}=\frac{1}{\lambda}\frac{w_{n/\lambda}^{(r-1)}}{\mu^{n/\lambda}}. \tag{4.2}$$

Now,

$$w_n^{(r)}=\sum_i f_i^{(r)}g_{n-i}^{(r)}=\mu^n\sum_i \frac{f_i^{(r)}}{\mu^i}\frac{g_{n-i}^{(r)}}{\mu^{n-i}}$$

$$=\mu^n\frac{1}{\lambda^2}\sum_i \frac{f_{i/\lambda}^{(r-1)}}{\mu^{i/\lambda}}\frac{g_{(n-i)/\lambda}^{(r-1)}}{\mu^{(n-i)/\lambda}}.$$

Substituting $j=i/\lambda$, we have

$$w_n^{(r)}=\mu^n\frac{1}{\lambda^2}\frac{1}{\mu^{n/\lambda}}\sum_{j=0/\lambda,1/\lambda\ldots} f_j^{(-1)}g_{n/\lambda-j}^{(-1)}.$$

Knowing that the polynomials are smooth, we can convert the sum to a sum over integers by keeping only 1/λ of the terms and multiplying by λ. Then,

$$w_n^{(r)}=\mu^n\frac{1}{\lambda}\frac{1}{\mu^{n/\lambda}}\sum_j f_j^{(r-1)}g_{n/\lambda-j}^{(r-1)}=\mu^n\frac{1}{\lambda}\frac{w_{n/\lambda}^{(r-1)}}{\mu^{n/\lambda}},$$

which proves Eq. (4.2). The proof of Eq. (4.1) follows directly. By induction, from Eq. (4.2), powers of a function that scale as in Eq. (2.3) also scale as in Eq. (2.3). Clearly, multiplication of a function by a constant maintains Eq. (2.3) and sums of products that satisfy Eq. (2.3) also satisfy Eq. (2.3). Therefore polynomials of functions that satisfy Eq. (2.3) also satisfy Eq. (2.3) and Eq. (4.1) is proven.

V. IMPROVED APPROXIMATION TO $F^{(r)}(x)$

We can use the coefficient scaling relation to derive a better approximation than the linear one to $F^{(r)}(x)$ near x_c. Using Eq. (2.3), we have

$$F^{(r)}(x)=\frac{1}{\lambda}\sum_n \frac{\mu^n}{\mu^{n/\lambda}}a_{n/\lambda}^{(r-1)}x^n.$$

Letting $m\equiv n/\lambda$, we have

$$F^{(r)}(x)=\frac{1}{\lambda}\sum_{m=0/\lambda,1/\lambda,2/\lambda,\ldots}\left(\frac{(\mu x)^\lambda}{\mu}\right)^m a_m^{(r-1)}.$$

For x near x_c, the summand is slowly varying and we can convert the sum to a sum over integers by keeping only 1/λ of the terms and multiplying by λ. We then have

$$F^{(r)}(x)=\sum_m\left(\frac{(\mu x)^\lambda}{\mu}\right)^m a_m^{(r-1)}=F^{(r-1)}\left(\frac{(\mu x)^\lambda}{\mu}\right).$$

Defining

$$f(x)=\frac{(\mu x)^\lambda}{\mu}, \tag{5.1}$$

we have by induction

$$F^{(r)}(x)=f^{(r)}(x)=\frac{(\mu x)^{\lambda r}}{\mu}. \tag{5.2}$$

At $x=x_c$, $f^{(r)}(x)=x_c$ as it should and the eigenvalue of this approximation is still λ. We note that if we desire a monomial approximation to $F(x)$, which has fixed point x_c and eigenvalue λ, our approximation is unique. Also, the approximation has derivatives as in Eq. (2.4). We also note that this approximation is not valid for $x<0$ where for noninteger λ the results are not real. This approximation is a significant improvement over the linear approximation as seen in Fig. 4.

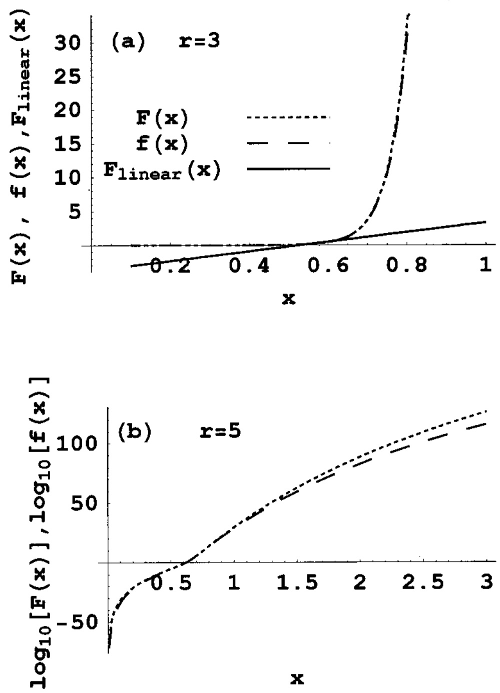

FIG. 4. Comparison of $F(x)=x^2+x^3$, its best monomial approximation $f(x)$, and its linear approximation F_{linear}, near $x=x_c$. As shown in (a), near x_c, $F(x)$ and $f(x)$ are basically coincident. In (b), for larger $|x-x_c|$, $F(x)$ and $f(x)$ diverge but $f(x)$ is still a significantly better approximation than F_{linear}, which would be coincident with the x axis, if plotted.

VI. APPLICATION TO CRITICAL PHENOMENA

A. Closed-form solution

We will apply coefficient scaling to the statistics of self-avoiding walks on a family of fractals. In particular, we will study the statistics of the average number of closed loops per site on the family of Sierpinski gaskets characterized by the length b of the side of the generating figure. Using renormalization-group theory, this family of fractals has been analyzed by Dhar [2] for $b=2$ and for higher b by Elezović *et al.* [3] to calculate critical points and critical exponents. We will use coefficient scaling to show how additionally the amplitudes can be calculated.

From [3], the average number of closed loops of length n per site, is given by the coefficients $p_{b,n}$ of $P_b(x)$, where

$$P_b(x)=\sum_{r=0}^{\infty}\frac{G_b(T_b^{(r)}(x))}{(k_b)^{r+1}}=\sum_{r=0}^{\infty}p_{b,n}x^n, \tag{6.1}$$

where $k_b=b(b+1)/2$, G_b is a polynomial of degree k_b, T_b is the renormalization-group transformation function for the system and is a polynomial of degree k_b with fixed point

$x_{c,b}$, connectivity constant $\mu_b=1/x_{c,b}$ and eigenvalue λ_b, and $T_b^{(r)}=T_b(T_b^{(r-1)})$. For simplicity, the b subscripts will be understood in what follows.

We then have

$$p_n=\left(\sum_{r=0}^{\infty}\frac{G(T^{(r)}(x))}{k^{r+1}}\right)_n=\sum_{r=0}^{\infty}\frac{(G(T^{(r)}(x)))_n}{k^{r+1}}, \quad (6.2)$$

where $(F(x))_n$ denotes the nth coefficient of $F(x)$.

Choosing an integer, $r_0>1$, such that Eq. (2.8) holds, we separate the sum into two parts:

$$p_n=\sum_{r=0}^{r_0-1}\frac{(G(T^{(r)}))_n}{k^{r+1}}+\sum_{r=r_0}^{\infty}\frac{(G(T^{(r)}))_n}{k^{r+1}}.$$

Since G and T are polynomials of degree k, the maximum n to which the first sum can contribute is $n_0=k^{r_0}$. Thus, for $n>n_0$, we can drop the first sum. Then, defining

$$\overline{G_n}=\frac{G_n}{\mu^n}, \quad (6.3a)$$

$$\overline{p_n}=\frac{p_n}{\mu^n}, \quad (6.3b)$$

and using Eq. (2.8), we have

$$\overline{p_n}=\sum_{r=r_0}^{\infty}\left(\frac{1}{\lambda}\right)^{r-r_0}\frac{(\overline{G}(T^{(r_0)}))_{n/\lambda^{r-r_0}}}{k^{r+1}}.$$

We will follow the approach of Derrida, DeSeze, and Itzytson [4] in analyzing a function of the form of Eq. (6.2) but we will analyze the coefficients as opposed to the function. Thus we perform a Mellin transform on $\overline{p_n}$ yielding

$$\begin{aligned}m(s)&=\int_0^{\infty}n^{s-1}\overline{p_n}dn\\&=\int_0^{\infty}n^{s-1}\sum_{r=0}^{\infty}\left(\frac{1}{\lambda}\right)^{r-r_0}\frac{(\overline{G}(T^{(r_0)}))_{n/\lambda^{r-r_0}}}{k^{r+1}}dn.\end{aligned}$$

Substituting n for n/λ^{r-r_0} for each r we have,

$$m(s)=(\lambda^{r_0})^{1-s}\int_0^{\infty}(\overline{G}(T^{(r_0)}))_n n^{s-1}\sum_{r=r_0}^{\infty}\frac{(\lambda^{s-1})^r}{k^{r+1}}dn. \quad (6.4)$$

The sum can be performed exactly and we have

$$m(s)=\frac{1}{k^{r_0-1}}\int_0^{\infty}\frac{(\overline{G}(T^{(r_0)}))_n}{k-\lambda^{s-1}}n^{s-1}dn=\frac{1}{k^{r_0-1}}\frac{\tilde{\tilde{G}}(s)}{k-\lambda^{s-1}},$$

where we've defined

$$\tilde{\tilde{G}}(s)=\int_0^{\infty}(\overline{G}(T^{(r_0)}))_n n^{s-1}dn. \quad (6.5)$$

Taking the inverse Mellin transform, we have

$$\overline{p_n}=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}m(s)\frac{1}{n^s}ds=\frac{1}{2\pi i k^{r_0-1}}\int_{-i\infty}^{+i\infty}\frac{\tilde{\tilde{G}}(s)}{k-\lambda^{s-1}}ds.$$

The integrand has poles at

$$s=\frac{\ln(k)}{\ln(\lambda)}+1+\frac{2\pi i m}{\ln(\lambda)}=a+mbi, \quad m=0,\pm1,\pm2,\pm3,$$

where

$$a\equiv\frac{\ln(k)}{\ln(\lambda)}+1, \quad c\equiv\frac{2\pi}{\ln(\lambda)}.$$

Using the residue theorem to evaluate the integral and using Eq. (6.3b), we have

$$\begin{aligned}p_n&=\mu^n\sum_{m=-\infty}^{\infty}\frac{1}{n^{(a+mci)}}\frac{\tilde{\tilde{G}}(a+mci)}{k^{r_0-1}k\ln(\lambda)}\\&=\mu^n n^{-a}\sum_{m=-\infty}^{\infty}e^{-2\pi im[\ln(n)/\ln(\lambda)]}\frac{\tilde{\tilde{G}}(a+mci)}{k^{r_0-1}k\ln(\lambda)}. \quad (6.6)\end{aligned}$$

The amplitudes of the Fourier contributions to the normalized coefficients are then

$$A_m=\frac{\tilde{\tilde{G}}(a+mci)}{k^{r_0-1}k\ln[\lambda]}. \quad (6.7)$$

The conventional representation for p_n is

$$p_n=\mu^n n^{\alpha-3}(\text{constant}+\text{correction terms}).$$

Comparing with Eq. (6.6), we find $\alpha=2-\ln(k)\ln(\lambda)$, which is consistent with earlier work [2,3]. In Appendix C, we derive the formula corresponding to Eq. (6.6) for $P(x)$.

We note that, while we are addressing the statistics of self-avoiding walks on fractals, Eqs. (6.6) and (6.7) apply to any observable represented by Eq. (6.1), as long as $\tilde{\tilde{G}}(s)$ is analytic within the contour of integration. If it is not, Eqs. (6.6) and (6.7) must be modified to take this into account. For the system we are studying, since G and T are polynomials, $\tilde{\tilde{G}}(s)$ is well behaved in the contour of integration.

B. Theoretical values for the amplitudes

The theoretical values of the amplitudes, Eq. (6.7), are obtained by calculating $(T^{(r_0)})_n$ recursively, calculating the coefficients of $\overline{G}(T^{(r_0)})$, creating an interpolated function from the coefficients, and then integrating the interpolated function in accordance with Eq. (6.5). The results are shown in Table I.

C. Numerical results

Numerical results were obtained by recursively calculating the coefficients of $T^{(r)}$ and then calculating the coefficients of $\overline{G}(T^{(r)})$ for each r and summing the results. The calculations were performed on a personal computer with a 233-MHz Pentium processor and 32 megabytes of memory and were accomplished on the order of minutes to hours. We obtained numerical results for $\overline{p_n}$ for $b=2-4$. The results are

TABLE I. Theoretical and measured values for the amplitudes A_m of the Fourier transform components of p_n, the average number of closed-loop self-avoiding walks on the Sierpinski gasket with length of side b, for $b=2-4$. The value of r used in the theoretical calculation is r_0; r_m is the number of iterations that were used to generate the sequence of self-avoiding walks, the amplitudes of which were measured.

b	Result type	r_0 r_m	Longest walk	Amplitudes A_0	A_1	A_2	A_3	A_4	A_5
2	Theory	$r_0=4$	81	.3656	.2036	.030 67	.002 052	.000 7960	.000 287 5
	measured	$r_m=14$	49 152	.3657	.1951	.026 59	.001 457	.000 3873	.000 103 6
3	Theory	$r_0=3$	216	.4339	.1312	.1447	.060 02	.007 753	.004 958
	measured	$r_m=9$	59 049	.4180	.1236	.1319	.051 50	.006 985	.004 213
4	Theory	$r_0=2$	100	.4661	.1413	.1185	.071 72	.063 28	.024 58
	measured	$r_m=6$	12 288	.4304	.1244	.1050	.062 94	.046 26	.015 65

shown in Figs. 5 and 6. Note the well-defined periodic oscillations and the subminima and submaxima indicating significant higher harmonics for $b=3-4$. To obtain the amplitudes, we then created an interpolated function from $\overline{p_n}$ and performed a Fourier transform on the last period of the function. The results are shown in Table I.

D. Comparison of theory with numerical results

Table I compares the theoretical values obtained using the lowest reasonable r_0 with the values obtained from the longest calculation of p_n. We see that we get good agreement between the theoretical values and numerical values for the amplitudes even though r_0 is considerably lower than the value of r used to calculate the amplitudes numerically. Also we see that, while the amplitudes drop off in magnitude quickly for $b=2$, higher Fourier components are significant for $b=3$ and 4, where the higher harmonics are present.

VII. MATHEMATICAL ORIGINS OF CRITICAL PHENOMENA

If we review the derivation of Eqs. (6.6) and (6.7) we realize that the general form depended on two facts: (a) the well-known form for the sum of a geometric series

$$\sum_{n=0}^{m} a^n = \frac{1-a^{m+1}}{1-a}$$

used to evaluate Eq. (6.4), which led to the poles in $m(s)$ and the corresponding exponential behavior, and (b) the scaling of the polynomial coefficients, which allowed the sum in Eq. (6.2) to be converted into a geometric series and which is the source of the μ^n factor.

Thus, at least for systems which have polynomial renormalization-group transformations and physical observables that are, or can be approximated by, sums of polynomials of iterations of the renormalization-group transformation, we can now understand that the critical behavior is directly related to the recursion relations for the coefficients of the iterated polynomial. Thus, for the class of systems we have described, critical behavior is a direct manifestation of the mathematical properties of iterated polynomials.

We believe that recursion relations similar to those for polynomials hold for the coefficients of the numerator and denominator in rational functions. If true, we would then have an understanding of the origin of critical behavior in a very large class of physical systems. A proof of this is in progress.

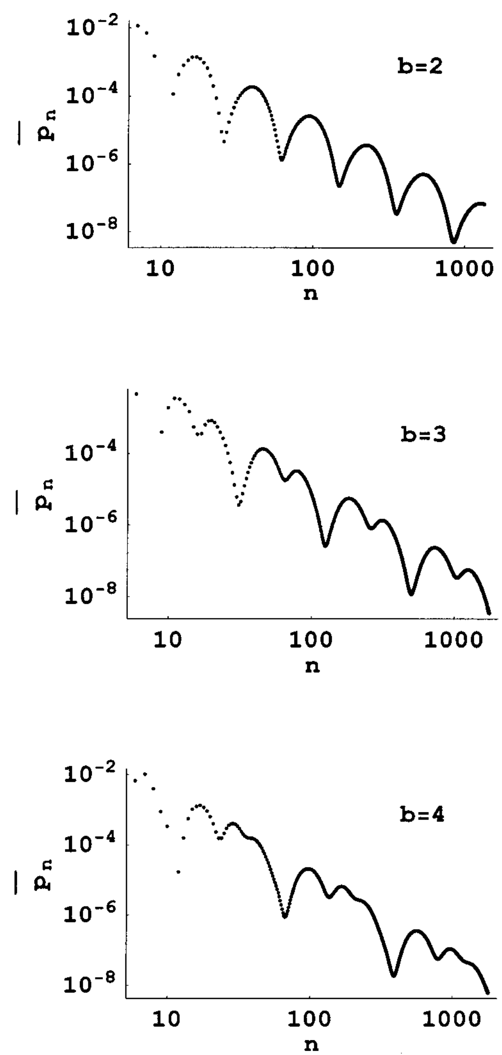

FIG. 5. The normalized coefficients $\overline{p_n}$ for $b=2-4$. The $\overline{p_n}$ represent the average number of closed loops per site of length n divided by μ^n.

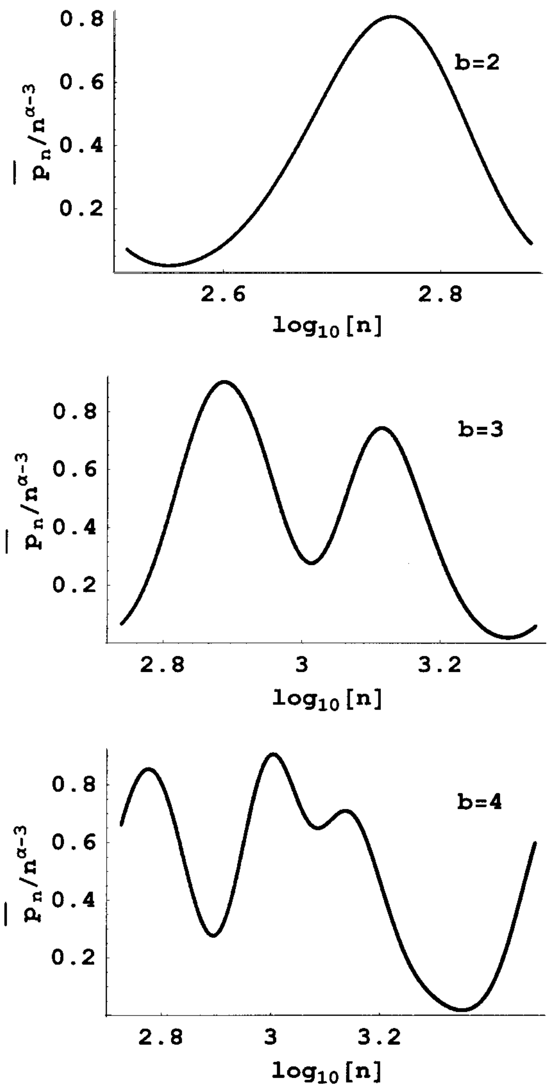

FIG. 6. The normalized coefficients $\overline{p_n}$ divided by $n^{\alpha-3}$, for $b=2-4$. The plots are for a single period. By dividing by the exponential term we can see clearly the details of the log-periodic behavior.

VIII. SUMMARY

We have proven a recursion relation for the coefficients of a class of polynomials of physical interest. Using that relation, we have developed a theory to calculate the number of closed-loop self-avoiding walks on the Sierpinski gasket family of fractal lattices and found good agreement between the measured values of the amplitudes and those predicted by the theory. Finally, we have discussed how, for a class of systems, the characteristics of their critical behavior can be tied directly to the behavior of the polynomial coefficients as represented by the recursion relation.

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APPENDIX A: RATE OF CONVERGENCE

An important issue is the rate of convergence of the recursion relation. From the proof it is clear that the recursion relation will converge more quickly the greater the value of λ. How the value of λ depends on the other fixed points of the polynomial becomes clear if we represent the polynomial as $f(x)=g(x)+x$; the roots of $g(x)$ are the fixed points of $f(x)$. Representing $g(x)$ as the product of its roots,

$$g(x)=(x-x_c)(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n),$$

we can then represent the eigenvalue at x_c as

$$\lambda=\left.\frac{df(x)}{dx}\right|_{x_c}=\left.\frac{dg(x)}{dx}\right|_{x_c}+1$$
$$=(x_c-x_1)(x_c-x_2)(x_c-x_3)\ldots(x_c-x_n)+1.$$

If there is more than one root of $g(x)$ at x_c, then $\lambda=1$ and the recursion relation is not true. It is then interesting to consider polynomials the fixed points of which approach the fixed point that represents the critical point. Consider the polynomial

$$f(x)=\frac{x^{2n+1}+x^{n+1}}{2},$$

which has n fixed points (including $x=1$) on the unit circle, n complex roots on the circle $|x|=2$, and one root at $x=0$. One might expect that as n increases and other fixed points approach the fixed point at $x=1$ arbitrarily closely, that λ would approach 1. In fact $\lambda=(3n+2)/2$ and increases as the number of fixed points increase; so closeness of other fixed points to the critical point does not in itself imply slow convergence. Figure 7 illustrates the fast rate of convergence for $n=4$ ($\lambda=7$).

A polynomial, which does exhibit slow convergence when δ is small, is

$$f(x)=\frac{(1+\delta)+x^2}{2+\delta},$$

which has fixed points at $x=1$ and $x=1+\delta$. Choosing the critical point to be $x_c=1+\delta$, we find the corresponding eigenvalue at x_c,

$$\lambda=\frac{2(1+\delta)}{2+\delta}.$$

Figure 8 illustrates the slow convergence in the case $\delta=0.1$ ($\lambda=1.04762$).

APPENDIX B: SUM RULE PROOF

Let F be a rational function,

$$F(x)=\frac{\sum_n a_n x^n}{\sum_m b_m x^n},$$

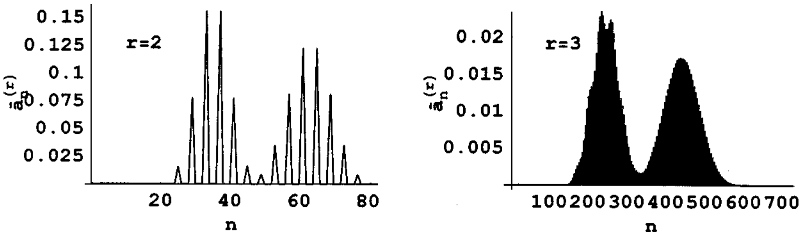

FIG. 7. The normalized coefficients $\overline{a_n^{(r)}}$ for iterations of $F(x)=(x^5+x^9)/2$ for $r=2,3$. Note the very fast convergence ($\lambda=7$).

where

$$F(x_c)=x_c .$$

Define $S(F)$ as the ratio of the sum of the normalized coefficients of F to the sum of the normalized coefficients of the denominator of F. That is,

$$S(F)\equiv\frac{\sum_n a_n x_c^n}{\sum_m b_m x_c^m}.$$

Then we will show that, if

$$F^{(r)}(x)=\frac{\sum_n a_n^{(r)} x^n}{\sum_m b_m^{(r)} x^m},$$

then

$$S(F^{(r)})=x_c .$$

The proof follows directly from the observation that S of any rational function is simply the function evaluated at $x=x_c$. Then,

$$S(F(F))=\frac{\sum_n a_n (F(x_c))^n}{\sum_m b_m (F(x_c))^m}=\frac{\sum_n a_n x_c^n}{\sum_m b_m x_c^m}=x_c .$$

The fact that $S(F^{(r)})=x_c$ follows by induction.

APPENDIX C: DERIVATION OF $P(x)$

Here we show that generally if

$$p_n=\mu^n n^a f\left(\frac{\ln(n)}{\ln(\lambda)}\right),$$

where f has period one, then for x less than and close to $1/\mu$,

$$P(x)\equiv\sum_{n=0}^{\infty} p_n x^n=\frac{f\left(\frac{\ln(1-\mu x)}{\ln(\lambda)}\right)}{(1-\mu x)^{1-a}}. \tag{C1}$$

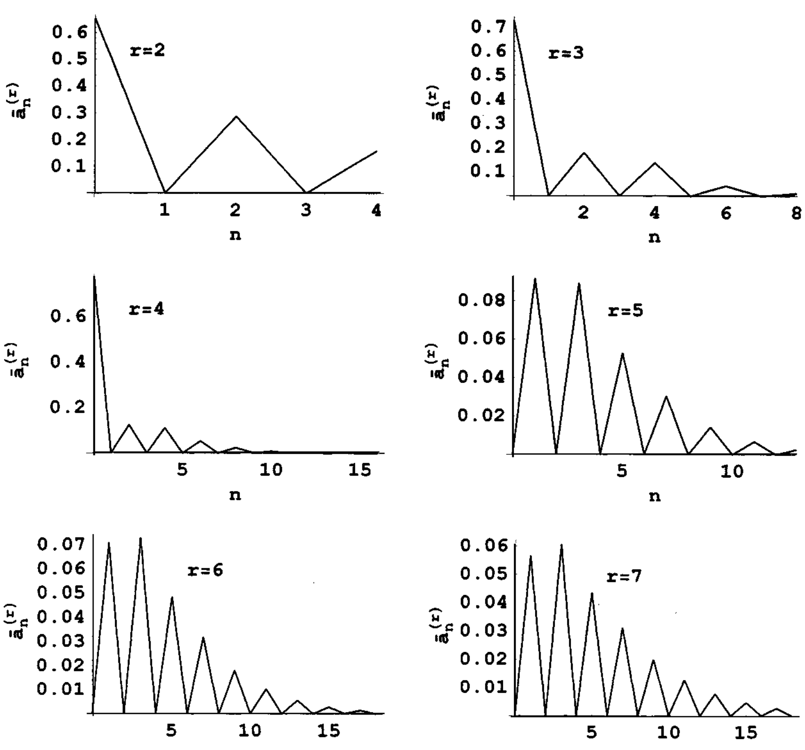

FIG. 8. The normalized coefficients $\overline{a_n^{(r)}}$ for iterations of $F(x)=(1+x^2)/4$ for $r=2-7$. Note the very slow convergence ($\lambda=1.05$).

Substituting in Eq. (C1) and expanding f in a Fourier series, we have

$$P(x)=\sum_{n=0}^{\infty}(\mu x)^n n^a \sum_{m=-\infty}^{\infty} a_m e^{2\pi i m \frac{\ln(n)}{\ln(\lambda)}}$$
$$=\sum_{n=0}^{\infty}(\mu x)^n n^a \sum_{m=-\infty}^{\infty} a_m n^{[2\pi i m/\ln(\lambda)]}$$
$$=\sum_{m=-\infty}^{\infty} a_m \sum_{n=0}^{\infty}(\mu x)^n n^{a+[2\pi i m/\ln(\lambda)]}.$$

For x near $1/\mu$, keeping the leading singular term, we have

$$P(x)=\sum_{m=-\infty}^{\infty} a_m \frac{1}{(1-\mu x)^{1-\{a+[2\pi i m/\ln(\lambda)]\}}}$$
$$=\sum_{m=-\infty}^{\infty} a_m \frac{(1-\mu x)^{[2\pi i m/\ln(\lambda)]}}{(1-\mu x)^{1-a}}$$
$$=\frac{1}{(1-\mu x)^{1-a}}\sum_{m=-\infty}^{\infty} a_m e^{[2\pi i m \ln(1-\mu x)/\ln(\lambda)]}$$
$$=\frac{f\left(\frac{\ln(1-\mu x)}{\ln(\lambda)}\right)}{(1-\mu x)^{1-a}}.$$

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